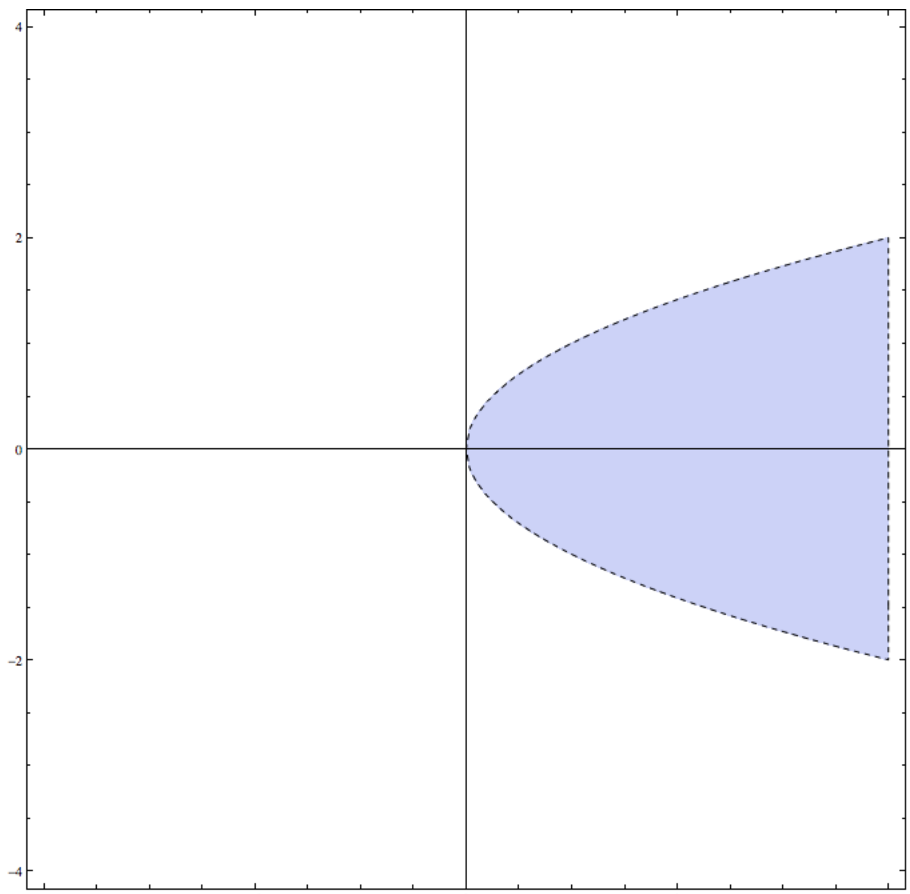
1. [25pts] Let $f(x, y) = \ln(x - y^2)$.

(a) Find the domain and range of $f(x, y)$.

Since the domain of $\ln(t)$ is all values of $t > 0$, we have:

$$\begin{aligned} Domain &= \{(x, y) \in \mathbb{R}^2 : x - y^2 > 0\} \\ &= \{(x, y) \in \mathbb{R}^2 : x > y^2\} \end{aligned}$$

While it wasn't necessary to sketch the domain in the xy-plane, here is a diagram showing the region. Note, the domain is shaded in blue, with an open (dashed) boundary, since ln is undefined for zero.



For the range, note that when $y = 0$, we have $f(x, 0) = \ln(x)$, and since $(x, 0)$ is in our domain for all positive values of x, we must have that the range of $f(x,y)$ is the same as the range of $\ln(x)$. That is:

$$Range = \{c \in \mathbb{R} : -\infty < c < \infty\},$$

i.e. the interval $(-\infty, \infty)$.

(b) Determine the equations of the level curves $f(x,y) = c$ together with possible values of c.

The possible values of c are exactly the values in the range. Hence $-\infty < c < \infty$. Setting $f(x,y) = c$ gives us:

$$\begin{aligned} \ln(x - y^2) &= c \\ x - y^2 &= e^c \\ x &= y^2 + e^c \end{aligned}$$

Since $0 < e^c < \infty$ for $-\infty < c < \infty$, then the level curves are exactly copies of the (sideways) parabola $x = y^2$, shifted to the right by the positive value e^c.

(c) Find a unit vector $\vec{u}$ which is perpendicular to the level curve of $f(x,y)$ passing through the point $(4,1)$.

The gradient vector $\nabla f(x_0, y_0)$ will always be perpendicular to the level curve that passes through the point (x_0, y_0):

$$\begin{aligned} \nabla f(4,1) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\Bigg|_{(4,1)} \\ &= \left(\frac{1}{x - y^2}, \frac{-2y}{x - y^2}\right)\Bigg|_{(4,1)} \\ &= (1/3, -2/3). \end{aligned}$$

Hence a unit vector in the same direction as $\nabla f(x,y)$ will also be perpendicular to the level curve through $(4,1)$:

$$\begin{aligned} \vec{u} &= \frac{\nabla f(4,1)}{||\nabla f(4,1)||} \\ &= \frac{3}{\sqrt{5}}(1/3, -2/3) = \frac{1}{\sqrt{5}}(1, -2) \end{aligned}$$

2. [20 points] Let $f(x,y) = e^{x^2}\sin(xy)$ be a function of two variables.

(a) Find the linearization of $f(x,y)$ at the point $(1,0)$.

$$f(1,0) = 0$$

$$\begin{aligned}\left.\frac{\partial f}{\partial x}\right|_{(1,0)} &= \left. 2xe^{x^2}\sin(xy) + e^{x^2}y\cos(xy)\right|_{(1,0)} \\ &= \left. e^{x^2}(2x\sin(xy) + y\cos(xy))\right|_{(1,0)} \\ &= 0\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial f}{\partial y}\right|_{(1,0)} &= \left. xe^{x^2}\cos(xy)\right|_{(1,0)} \\ &= e\end{aligned}$$

Hence the linearization of $f(x,y)$ at $(1,0)$ is:

$$\begin{aligned}L(x,y) &= f(1,0) + \left.\frac{\partial f}{\partial x}\right|_{(1,0)}(x-1) + \left.\frac{\partial f}{\partial y}\right|_{(1,0)}(y-0) \\ &= 0 + (0)(x-1) + (e)(y-0) \\ &= ey\end{aligned}$$

(b) Use your answer in part (a) to approximate $f(1.1, -0.05)$.

$$\begin{aligned}f(1.1,-0.05) &\approx L(1.1,-0.05) \\ &= -0.05e,\end{aligned}$$

which is about $2.718 \times -0.05 = -0.1359$. (The actual value is $-0.18434868...$).

3. [15 points] Let $w = \sin(f(x,y))$ be a function of x, y and $x = u(t),\ y = v(t)$ be functions of t. Find $\frac{dw}{dt}$ in terms of $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, u, v, u', v'$.

$$\begin{aligned}
\frac{dw}{dt} &= \frac{d}{dt} sin(f(u(t), v(t))) \\
&= \cos(f(u(t), v(t)))\ \frac{d}{dt} f(u(t), v(t)) \\
&= \cos(f(u(t), v(t)))\ \left(\left.\frac{\partial f}{\partial x}\right|_{(u(t),v(t))} \frac{dx}{dt} + \left.\frac{\partial f}{\partial y}\right|_{(u(t),v(t))} \frac{dy}{dt} \right) \\
&= \Big(\cos(f(u(t), v(t))) \Big) \left(\frac{\partial f}{\partial x}(u(t), v(t))\ u'(t) + \frac{\partial f}{\partial y}(u(t), v(t))\ v'(t) \right)
\end{aligned}$$

4. [20 points] Find absolute minimum and maximum of the function $f(x,y) = 12xy - 4x^2y - 3xy^2$ on the triangle bounded by the x-axis, y-axis and the line $4x + 3y = 12$.

i.

$$\begin{aligned}\nabla f(x,y) &= (12y - 8xy - 3y^2, 12x - 4x^2 - 6xy) \\ &= (y(12 - 8x - 3y), x(12 - 4x - 6y))\end{aligned}$$

Setting $\nabla f(x,y) = (0,0)$ gives us

$$\begin{aligned}y(12 - 8x - 3y) &= 0 \\ x(12 - 4x - 6y) &= 0.\end{aligned}$$

The first equation gives us $y = 0$ or $12 - 8x - 3y = 0$. Letting $y = 0$, and substituting into the second equation gives us

$$\begin{aligned}x(12 - 4x - 3(0)) &= 0 \\ \implies 4x(3 - x) &= 0 \\ \implies x = 0 \text{ or } x &= 3.\end{aligned}$$

Hence $(0,0)$ and $(3,0)$ are two critical points (and candidates for minima/maxima).

Letting $12 - 8x - 3y = 0$ gives us $y = 4 - \frac{8}{3}x$. Substituting into the second equation gives us

$$\begin{aligned}x(12 - 4x - 6(4 - \tfrac{8}{3}x)) &= 0 \\ \implies 12x(x - 1) &= 0 \\ \implies x = 0 \text{ or } x &= 1 \\ \implies y = 4 \text{ or } y &= \tfrac{4}{3}.\end{aligned}$$

Hence $(0,4)$ and $(1, \frac{4}{3})$ are two critical points (and candidates for maxima/minima).

ii. Along the x-axis we have $f(x,y) = f(x,0)$, with

$$\begin{aligned}f(x,0) &= 12x(0) - 4x^2(0) - 3x(0)^2 \\ &= 0.\end{aligned}$$

Hence $\frac{d}{dx} f(x,0) = 0$ everywhere along the x-axis, and so all points $(x,0)$ are critical points along the boundary (and candidates for maxima/minima).

Along the y-axis we have $f(x,y) = f(0,y)$, with

$$\begin{aligned} f(0,y) &= 12(0)y - 4(0)^2y - 3(0)y^2 \\ &= 0. \end{aligned}$$

Hence $\frac{d}{dy} f(0,y) = 0$ everywhere along the y-axis, and so all points $(0,y)$ are critical points along the boundary (and candidates for maxima/minima).

Along the line $4x+3y = 12$ we have $y = 4-\frac{4}{3}x$, and $f(x,y) = f(x, 4-\frac{4}{3}x)$, with

$$\begin{aligned} f(x, 4 - \tfrac{4}{3}x) &= 12x(4 - \tfrac{4}{3}x) - 4x^2(4 - \tfrac{4}{3}x) - 3x(4 - \tfrac{4}{3}x)^2 \\ &= 0. \end{aligned}$$

Hence $\frac{d}{dx} f(x, 4 - \frac{4}{3}x) = 0$ everywhere along the line $4x + 3y = 12$, and so all points $(x, 4 - \frac{4}{3}x)$ are critical points along the boundary (and candidates for maxima/minima).

iii. Finally, we must test all the critical points both in the domain and along the boundary:

$$\begin{aligned} f(0,0) &= 0 \\ f(3,0) &= 0 \\ f(0,4) &= 0 \\ f(1,\tfrac{4}{3}) &= \tfrac{16}{3} \\ f(x,0) &= 0 \\ f(0,y) &= 0 \\ f(x, 4 - \tfrac{4}{3}x) &= 0 \end{aligned}$$

Hence there is an absolute maximum of $\frac{16}{3}$ at the point $(1, \frac{4}{3})$, and an absolute minimum of 0 along the entire boundary.

5. [20 points] Given the system

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = 2y$$

(a) **Write the system in the matrix form** $\frac{d\vec{x}}{dt} = A\vec{x}(t)$ **Find all eigenvalues and corresponding eigenvectors of** A.

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 = 0$$

Hence $\lambda = 2$ is the only eigenvalue of A. To find the corresponding eigenvector,

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 2 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$\implies 2u_1 + u_2 = 2u_1$$
$$\implies u_2 = 0.$$

Hence $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 2.

(b) **Show that** $\vec{x}(t) = e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + te^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **is solution of the system.**

$$e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + te^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} te^{2t} \\ e^{2t} \end{bmatrix}$$

Hence

$$\begin{aligned}
\frac{d\vec{x}}{dt} = \begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} &= \begin{bmatrix} e^{2t} + t(2e^{2t}) \\ \\ 2e^{2t} \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} te^{2t} \\ \\ e^{2t} \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ \\ y(t) \end{bmatrix}.
\end{aligned}$$

Hence this is indeed a solution of the (matrix) differential equation.